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THE FLOATING BEHAVIOUR OF A SMALL BODY ACTED UPON BY A SURFACE WAVE†

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The passive motion of a body on the ruffled surface of a liquid acted upon by gravity and drag forces is considered. The dimensions of the body are much less than the wavelength and the oscillations of the body in a direction normal to the liquid surface are ignored. Two cases of motion, corresponding to turbulent flow around the body and gliding, are investigated. © 1999 Elsevier Science Ltd. All rights reserved.

An investigation of the possibility of translational passive motion of a floating body when there is a surface swell showed [1] that radiation wave pressure exists due to reflection of waves from the body. This pressure is considerable when the dimensions of the body are comparable with or greater than the wavelength. If the dimensions of the body are much less than the wavelength, the action of the body on the wave can be neglected. In this case the resultant of the projections of the gravity force and the reaction force of the liquid onto the plane tangential to the deformed surface of the liquid may be non-zero, and hence the body will move over the liquid surface.

Hence, under natural conditions, wave drift of floating ice occurs. A theoretical investigation of the drift of small ice floes in a surface-wave field of small amplitude showed [2, 3] that the effect of the waves leads to inelastic collisions between the ice floes and to the formation from them of fields of pack ice, which are stable due to radiation pressure of the waves [4]. The swell may cause the ice floes to accelerate in the coastal area and damage hydro-technical structures and the sea bottom (the phenomenon of an ice storm [5]).

It is of interest to investigate the acceleration of small bodies on surface waves of large amplitude. This effect is most pronounced near the coast at shallow depths, since as a wave approaches the shore its profile becomes steeper and the projection of the gravity force onto the tangential plane to the liquid surface increases. Another reason for the acceleration of small floating bodies by waves is the fact that, when there is a gradual reduction in the depth, the wave profile approaches a limit Stokes wave [6], which has a corner point with an angular aperture of 120° on the crest. The velocity of a liquid particle at the corner point is equal to the velocity of the wave. Hence, it becomes possible for small bodies to become captured on the wave crest, and the body moves with the wave velocity and the force acting on its from the liquid side is zero.

The propagation velocity of a wave on shallow water may reach several metres per second. At such a velocity small bodies of streamline shape will glide [7]. Under these conditions the main part of the drag will be due to the formation of spray jets of water. A typical example of rapid motion on a wave in the coastal region, when the flow may be turbulent or gliding can occur, depending on the velocity of the body, is surfing.

In [8], using numerical methods, a steady solution of the equation of motion of an ideal incompressible liquid was constructed in which part of the surface streamline has zero curvature and is a section. This part may correspond to the surface of a floating body when the forces and moments acting on the body balance; the conditions under which this balance occurs have not been investigated.

Below we consider different forms of motion of a small floating body on a steady surface wave. The condition for the body to be captured by the wave when it moves subject to a square drag law and when gliding are investigated.

1. BASIC EQUATIONS

Suppose the surface of the liquid is deformed by a plane steady surface wave, propagating with constant velocity c in a horizontal direction x, and the dimensions of the body are small compared with the

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wavelength. Rotational motions of the body are ignored, and it is in fact a point mass, sliding over the liquid surface. AT the same time, the drag of the liquid R acting on the body depends on its shape.

We will decompose the drag into components normal and tangential to the liquid surface: $\mathbf{R} = R_n\mathbf{n} + R_\tau\boldsymbol{\tau}$, where $\mathbf{n}$ and $\boldsymbol{\tau}$ are the unit vectors normal and tangential to the liquid surface. We will assume that at each instant of time the tangents to the body trajectory and to the liquid surface at the corresponding point are parallel. The body is not displaced in the direction $\mathbf{n}$. Hence we have the relations

$$\mathbf{v}\cdot\mathbf{n} = 0, \quad \mathbf{v}_w\cdot\mathbf{n} = 0 \tag{1.1}$$

$$\mathbf{n} = \frac{(-\eta_\theta, 1)}{\sqrt{1+\eta_\theta^2}}, \quad \tau = \frac{(1, \eta_\theta)}{\sqrt{1+\eta_\theta^2}}, \quad \eta_\theta \equiv \frac{\partial\eta}{\partial\theta}$$

where $\mathbf{v}$ is the body velocity, $\mathbf{v}_w$ is the velocity of the liquid particles on the wave surface, c is the wave velocity and $\eta(\theta)$ is the deviation of the liquid surface from the horizontal equilibrium position.

The equations of the body motion in a system of coordinates moving with the wave have the form

$$m\left(\frac{dv}{dt}\boldsymbol{\tau} + \frac{v^2}{r}\frac{\eta_{\theta\theta}}{|\eta_{\theta\theta}|}\mathbf{n}\right) = \mathbf{F}_g + \mathbf{R} \tag{1.2}$$

$$v = \mathbf{v}\cdot\boldsymbol{\tau}, \quad \frac{1}{r} = \frac{|\eta_{\theta\theta}|}{(1+\eta_\theta^2)^{3/2}}, \quad \eta_{\theta\theta} \equiv \frac{\partial^2\eta}{\partial\theta^2}$$

Here $\mathbf{F}_q = (0, -mg)$ is the gravity force and m is the body mass.

Multiplying Eq. (1.2) scalarly by $\mathbf{n}$ we obtain the balance condition for the centripetal force, the resistance of the liquid and the gravity force, projected onto the normal to the liquid surface

$$R_n = \frac{mg}{\sqrt{1+\eta_\theta^2}} + \frac{mv^2}{r}\frac{\eta_{\theta\theta}}{|\eta_{\theta\theta}|} \tag{1.3}$$

We will assume that the drag R_τ depends on the difference in the velocities $\upsilon - \upsilon_w$, where $u_w = \mathbf{v}_w\tau$, and vanishes when $\upsilon - \upsilon_w$

$$R_\tau = R_\tau(v - v_w, \alpha, \beta, \ldots), \quad R_\tau(0, \alpha, \beta, \ldots) = 0 \tag{1.4}$$

The parameters α, β, . . . are determined by the body shape and the liquid viscosity. In general, the force R_τ also depends on the derivatives with respect to time of $\upsilon - \upsilon_w$. This dependence is related to the transfer of part of the body momentum to the liquid when it accelerates and gives rise to the existence of added masses. For a body which is elongated in the direction of motion, the coefficients of the added masses are small, and we can neglect the dependence of R_τ on the derivatives.

Multiplying Eq. (1.2) scalarly by τ, we obtain the system of equations

$$m\frac{dv}{dt} = -\frac{mg\eta_\theta}{\sqrt{1+\eta_\theta^2}} + R_\tau, \quad \frac{d\theta}{dt} = \frac{v}{\sqrt{1+\eta_\theta^2}} \tag{1.5}$$

The second equation of (1.5) defines the horizontal component of the body velocity in a system of coordinates moving with the wave.

Assuming $\upsilon - \upsilon_w$, we obtain from (1.5), taking (1.4) into account

$$\tfrac{1}{2}v_w^2 + g\eta = \text{const} \tag{1.6}$$

Relation (1.6) is identical with the Bernoulli integral written in a moving system of coordinates. Hence it follows that Eqs (1.5) always have a solution which describes the oscillatory motion of the body for which the motion of a liquid particle in the wave.

2. THE BODY MOTION ON A WAVE FOR A QUADRATIC DRAG LAW

Suppose the drag on a body moving over the liquid surface is given by the formula

$$R_\tau = \frac{1}{2}\rho_w C_w S |v_w - v|(v_w - v) \tag{2.1}$$

where ρ_w is the liquid density, C_w is the drag coefficient and S is the wetted area of the body.

The drag relation (2.1) is due to turbulent friction and the dynamic pressure of the water on the wetted area of the body. The drag coefficient C_w depends on the Reynolds number and the shape of the wetted part of the body. Formula (2.1) is used when calculating the drag of ships [9] and in problems involving a calculation of the drift of floating ice [10]. Usually C_w varies from 10^{-3} to 10^{-2}. For poorly streamlined bodies the value of C_w may be of the order of unity.

The wetted area S depends on the depth to which the body is immersed, that depends on the normal reaction of the water R_n. An increase in R_n makes the body float up. When the velocity υ is fairly high the body may start to glide. This effect is ignored here.

It follows from (1.5) that steady motion of the body with the velocity of the wave is possible when the following condition is satisfied

$$\frac{mg\eta_\theta}{\sqrt{1+\eta_\theta^2}} = -\frac{1}{2}\rho_w C_w S v_w^2 \tag{2.2}$$

The quantities η and υ_w are functions of θ, and hence the possible values of the phase θ_j which define the body position on the wave under conditions of steady motion, are solutions of Eq. (2.2).

The velocity υ_w of the liquid particles on the surface of a wave of small amplitude is equal to, apart from small higher-order terms, the phase velocity of the wave in a system of coordinates connected with the wave, and is independent of the wave amplitude. Hence it follows from (2.2) that for steady motion of the body on the wave to occur the wave amplitude must be greater than a certain critical value. We will assume that the function $\eta(\theta)$ has two points of inflection between two neighbouring wave crests. Equation (2.2) then has two solutions in each period of the wave corresponding to a stable and unstable body position.

In Fig. 1(a) we show the form of the phase diagram of the solutions of the system of equations (1.5) in the case when the drag is given by (2.1) and Eq. (2.2) has two solutions per wave period. In the (θ, ν) phase plane there are stable for foci $F_{1,2}$ and saddles $S_{1,2}$. All the phase curves, apart from the two separatrices, are attracted to the foci or the periodic curve P, corresponding to oscillatory motion of the body, for which its velocity is equal to the velocity of the liquid particles. In Fig. 1(a) the separatrices, entering a saddle, are denoted $A_1S_1C_1S_1$ and A_2S_2, C_2S_2. The separatrices emerging from the saddle are represented by the curves S_1F_1, S_2F_2 and S_2G_2.

Some of the phase curves arriving from $+\infty$ with respect to υ and situated between separatrices A_1S_1 and C_1S_1 wind round the focus F_1. An example of curves of this type is curve B_1F_1. A similar assertion holds for he phase curves between separatrices A_2S_2 and C_2S_2.

The phase curves arriving from $+\infty$ with respect to υ and situated between separatrices C_1S_1 and A_2S_2 tend asymptotically to the periodic curve P. An example of this is curve D_1E_1. All the phase curves arriving from $-\infty$ with respect to υ tend asymptotically to curve P. Examples are curves P_1 and P_2.

It can be seen that in order for the body to accelerate the wave velocity it is necessary for the initial velocity of the body motion and the phase to be in the attraction region of some focus. For example, if the initial velocity of the body motion is υ_0, the body accelerates when the phase value lies in the range H_1I_1 or H_2I_2.

In Fig. 1(b) we show the form of the phase pattern of the solutions of systems of equations (1.5) when the body moves on the limiting Stokes waves. The velocity of the liquid particles at the corner points on the crests of the Stokes wave is equal to the wave velocity. Hence, the maxima of curve P coincide in this case with the saddles S_1 and S_2. All the phase curves arriving from $-\infty$ with respect to υ approach curve P asymptotically. All the phase curves, but the separatrices arriving from $+\infty$ with respect to υ, approach the foci asymptotically. Hence it can be seen that any small floating body is accelerated up to the wave velocity.

We will derive formulae for estimating the amplitudes of the linear waves for which steady motion of the body on the wave is possible. The wave parameters are completely defined by the velocity potential φ and the dispersion relation which relates the wave frequency ω to the wave number k

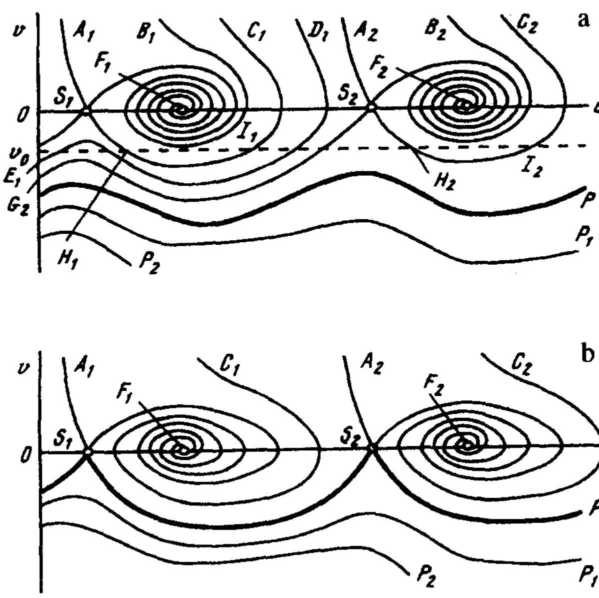

Fig. 1.

$$\varphi = \frac{\omega a}{k\,\mathrm{th}(kH)} \frac{\mathrm{ch}[k(z+H)]}{\mathrm{ch}(kH)} \sin(k\theta) \tag{2.3}$$

$$\omega^2 = gk\,\mathrm{th}(kH), \quad \varepsilon = ak \ll 1$$

(H is the liquid depth). The deviation of the liquid surface from the horizontal equilibrium position is $\eta = ak\cos(k\theta)$. The phase velocity of the wave is $c = \omega/k$. The horizontal velocity of the liquid particles in the wave in a fixed system of coordinates is $u = \partial\varphi/\partial x$.

The condition for a solution of Eq. (2.2) to exist has the form

$$\varepsilon > \varepsilon_*, \quad \varepsilon_* = \frac{\rho_w C_w}{\rho_i kh} \mathrm{th}(kH) \tag{2.4}$$

where ρ_i is the density of the floating body and h is a parameter representing its thickness.

It follows from (1.5) that

$$k\theta_1^n \approx 2n\pi + \arcsin\frac{\varepsilon_*}{\varepsilon}, \quad k\theta_2^n \approx (2n+1)\pi - \arcsin\frac{\varepsilon_*}{\varepsilon} \tag{2.5}$$

Linearizing Eqs (1.5) in the neighbourhood of the singular points, we obtain the eigenvalues

$$\lambda_1^\pm = -\omega\varepsilon_*\left[1 \pm \left(1 + \frac{1}{\varepsilon_*}\sqrt{\varepsilon^2 - \varepsilon_*^2}\right)^{1/2}\right], \quad \lambda_2^\pm = -\omega\varepsilon_*\left[1 \pm \left(1 - \frac{1}{\varepsilon_*}\sqrt{\varepsilon^2 - \varepsilon_*^2}\right)^{1/2}\right] \tag{2.6}$$

The eigenvalues $\lambda_1^\pm$ and $\lambda_2^\pm$ correspond to the singular points θ_1^n and θ_2^n. It follows from (2.6) that the eigenvalues $\lambda_1^\pm$ are real and have different signs. Hence, the points θ_1^n are saddle points. When $0 < (\varepsilon/\varepsilon_*)^2 - 1 < \varepsilon_*^2$ the eigenvalues $\lambda_2^\pm$ are real and negative. In this case the points θ_2^n are stable nodes. When $\varepsilon^2 < \varepsilon_*^2(1 + \varepsilon_*^2)$ the numbers $\lambda_2^\pm$ are complex conjugate quantities with a real part less than zero. In this case the points θ_2^n are stable foci.

It can be seen that the saddles are situated close to the wave crests and the nodes are close to the wave trough. In the limiting case when $\varepsilon = \varepsilon_*$ the saddles merge with the nodes at the point $\theta = (2n + 1/2)\pi$, where the wave has its maximum slope.

3. RESULTS OF NUMERICAL CALCULATIONS

Numerical calculations were carried out for the potential velocity field of liquid particles in the wave. The velocity potential φ and the elevation η of the liquid surface above the horizontal equilibrium position are given by the formulae [11]

$$\varphi = c_0\left[-\frac{B_0}{kH}x + \sum_{n=1}^{5} B_n \frac{\operatorname{ch}[nk(z+H)]}{\operatorname{sh}(nkH)}\sin kn\theta\right] \tag{3.1}$$

$$\eta = \frac{1}{k}\sum_{n=1}^{5} A_n \cos(nk\theta)$$

where the coefficients A_n, B_0 and B_n are fifth-order polynomials in the dimensionless parameter $\varepsilon = ka/2$, where a is the wave height, which is equal to the distance from a crest to a trough. The coefficients of the polynomials are rational expressions in $\operatorname{cth}(kH)$, where H is the depth of the undisturbed liquid. The parameter ε is assumed to be small.

The propagation velocity c of the non-linear wave (3.1) is given by the formula

$$c = c_0(1+\varepsilon^2\Delta_1+\varepsilon^4\Delta_2), \quad c_0 = \sqrt{(g/k)\operatorname{th}kH} \tag{3.2}$$

where Δ_1 and Δ_2 are rational expressions in kH and $\operatorname{cth}(kH)$.

Formulae (3.1) approximate the accurate solution of Euler's equations, corresponding to a periodic wave of height a and wavelength $2\pi/k$. The coefficients of the powers of the parameter ε increase without limit as $kH \to 0$. Hence for a specified wave height a, formulae (3.1) have a physical meaning when the value of the parameter kH is fairly large. When kH is reduced, to approximate the accurate solution it is necessary to take into account a larger number of terms of the asymptotic expansion in the parameter ε

In Fig. 2 we show the form of a wave height a = 1.2 m with wave number k = 0.1 m^{-1} for H = 5 m (curve 1) and H = 50 m (curve 2). It can be seen that as the depth decreases the slope of the wave

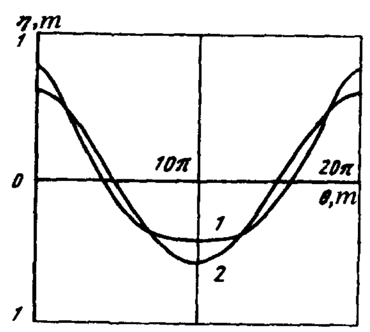

Fig. 2.

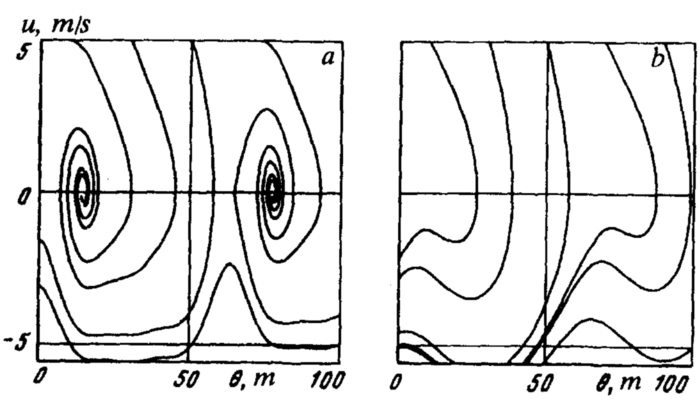

Fig. 3.

increases in the crest region and decreases in the trough region. The propagation velocities of waves 1 and 2 are approximately 7 m/s and 10 m/s.

In Fig. 3 we show integral curves of the motion of a floating body acted upon by a surface wave that are situated in the neighbourhood of the straight line $\upsilon = 0$ in a system of coordinates connected with the periodic wave. Cases a and b correspond to waves 1 and 2, shown in Fig. 2. Here $C_w = 0.004$, $\rho_i = 930$ kg/m^3 and $h = 0.3$ m. It can be seen that in case a the body can be captured by the wave if its initial velocity in a fixed system of coordinates is greater than 1 m/s. In case b there is no capture. Hence, when the wave arrives in shallow water its ability to capture small floating bodies increases.

4. THE MOTION OF A PLATE ON A WAVE IN THE GLIDING MODE

Consider the motion of a heavy plate on a wave in the gliding mode. The flow around the plate is shown in Fig. 4. It is assumed that the Froude number Fr > 2. The Froude number is calculated from the formula $\mathrm{Fr}^2 = gL/(\upsilon - \upsilon_w)^2$, where L is the plate width. In this case the drag R is given by the formula [7]

$$\mathbf{R} = P(-\sin(\beta-\gamma),\ \cos(\beta-\gamma)),\quad P = \rho_w \delta W(\upsilon - \upsilon_w)^2 \operatorname{ctg}\frac{\beta}{2},$$
$$\eta_\theta = -\operatorname{tg}\gamma \tag{4.1}$$

where δ is the width of the spray jet and W is the size of the plate in a direction perpendicular to the (θ, z) plane in Fig. 4.

From Eq. (1.6) we obtain

$$P\cos\beta = mg\cos\gamma + \frac{m\upsilon^2}{r}\frac{\eta_{\theta\theta}}{|\eta_{\theta\theta}|} \tag{4.2}$$

The system of equations (1.5) takes the form

$$m\frac{d\upsilon}{dt} = mg\sin\gamma - P\sin\beta,\quad \frac{d\theta}{dt} = \upsilon\cos\gamma \tag{4.3}$$

The moment M of the hydrodynamic force about the rear edge of the plate and the wetted length l are

$$M = PL^{-1}\delta^2 f(\beta),\quad l = \delta g(\beta) \tag{4.4}$$

(the functions $f(\beta)$ and $g(\beta)$ were derived earlier [7]).

Consider the conditions for steady motion of the plate on the wave when $\upsilon = 0$. In this case it follows from (4.2) and (4.3) that $\beta = \gamma$ and $P = mg$. In other words, the weight of the plate is equal to the pressure force, and the plate is horizontal.

The width of the spray jet is found from the second equation of (4.1), and the moment of the hydrodynamic forces is found from Eq. (4.4). Hence it follows that to maintain the steady motion of the plate a moment of the external forces must be applied to it which compensates the moment of the

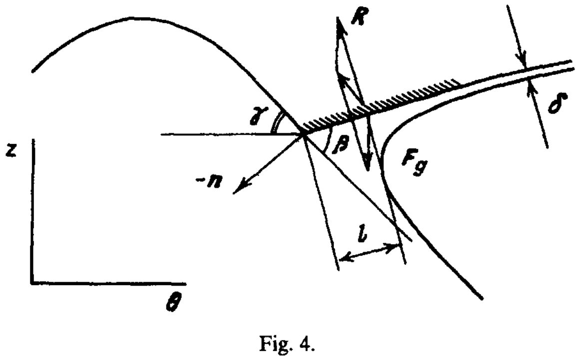

Fig. 4.

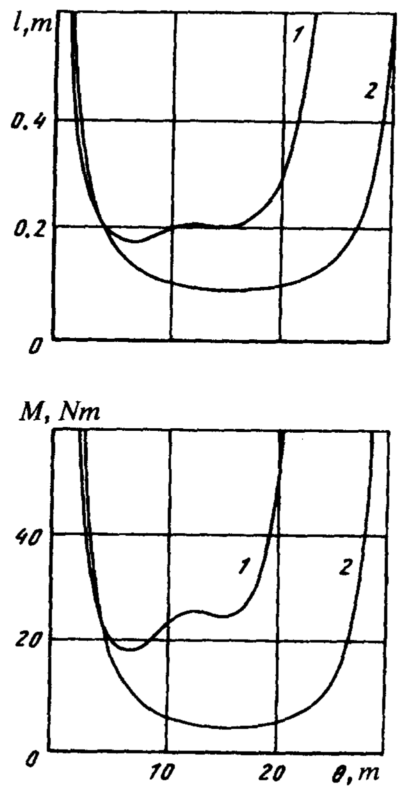

Fig. 5.

hydrodynamic forces. If the balance of the moments breaks down, the plate glides over the surface of the water.

In Fig. 5 we show the wetted length of the plate as a function of the wave phase for steady gliding on the wave when $m = 80$ kg and $W = 1$ m. Curves 1 and 2 correspond to the waves shown in Fig. 2. It can be seen that the best conditions for gliding on the wave occur in the region where the projection of the gravity force onto the wave surface is extremal. As one approaches the crest or the trough of the wave the moment increases without limit, and hence it is fairly difficult to maintain the plate in the steady state. Unlike the case considered in Section 3, the conditions for steady gliding on waves 1 and 2 differ only slightly.

5. CONCLUSION

Thus, in the case of turbulent flow around a body on a wave two modes of the body motion are asymptotically possible: a periodic motion and capture of the body by the wave. When moving under capture conditions the body velocity is equal to the phase velocity of the wave. The periodic mode of motion exists for any wave parameters. The capture mode only occurs when the wave amplitudes are sufficiently high. For the same height, waves in shallow water have a greater ability to capture floating bodies than waves on the surface of a deep liquid.

The practical realisation of the capture mode for typical parameters of sea swell is only possible in shallow water where the depth does not exceed 10 m. In this case a necessary initial velocity, exceeding the velocity of the surface liquid particles in its neighbourhood, must be imparted to the body.

The conditions for a heavy plate to be captured by a wave when gliding depend only slightly on the depth of the liquid. Such conditions can be realised for practical parameters of the sea swell.

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